Finite Element Methods for Numerical Computation of Flows in Porous Media

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Abstract

This paper starts control volume finite element methods which are locally conservative on each control volume. After that we consider potential- and flux-based upstream weighting strategies and the mixed finite element spaces on tetrahedra in three dimensions. Then various approaches such as linearization, implicit time approximation and explicit time approximation are studied. Finally, we discuss the application to one-phase flows and two-phase flows in porous media.

1 Control Volume Finite Element (CVFE) Methods

1.1 Basic CVFEs

To see the idea of CVFE, we focus on linear triangular elements in two dimensions. We consider the stationary problem

$$-\nabla \cdot (\mathbf{K} \nabla \mathrm{P}) = \mathrm{f} \qquad \text{in } \Omega, \tag{1}$$

where Ω is a bounded domain in the plane, $\mathbf{K}$ is a tensor and P is pressure.

Let V_i be a control volume. Replacing P by $P_h \in V_h$, the space of continuous piecewise linear functions on $\bar{\Omega}$, in (1) and integrating over V_i, we see that

$$-\int_{V_i} \nabla \cdot (\mathbf{K} \nabla P_h(x))\, dx = \int_{V_i} f(x)\, dx .$$

The divergence theorem implies

$$-\int_{\partial V_i} (\mathbf{K} \nabla P_h) \cdot \hat{\mathbf{n}}\, ds = \int_{V_i} f(x)\, dx, \tag{2}$$

where $\hat{\mathbf{n}}$ is the outward unit normal.

We note that $\nabla P_h \cdot \hat{\mathbf{n}}$ is continuous across each segment of ∂V_i. Thus, if $\mathbf{K}$ is continuous across that segment, so is the flux $(\mathbf{K} \nabla P_h) \cdot \hat{\mathbf{n}}$. Therefore, the flux is continuous across the edges of the control volume V_i. Furthermore, (2) indicates that the CVFE method is locally conservative.

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Given a triangle T with vertices m_i, m_j and m_k, edge midpoints m_a, m_b and m_c, and centre m_g, it follows that the approximation P_h to P on T is given by

$$P_h = P_i\beta_i + P_j\beta_j + P_k\beta_k, \tag{3}$$

where the local basis functions $\beta_i = \beta_i(x)$ satisfy

$$\beta_i(m_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

with

$$\beta_i(x) + \beta_j(x) + \beta_k(x) = 1, \qquad x \in T. \tag{4}$$

These basis functions are called the **barycentric coordinates** of the triangle T.

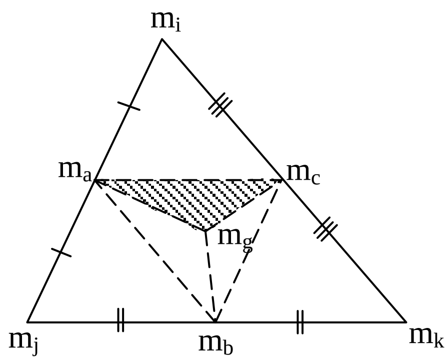

Figure 1: Triangle T

Set

$$\begin{aligned} a_i &= m_{j,2} - m_{k,2}, \\ b_i &= -(m_{j,1} - m_{k,1}), \\ g_i &= m_{j,1}\, m_{k,2} - m_{j,2}\, m_{k,1}, \end{aligned}$$

where $\mathbf{m}_i = [m_{i,1}, m_{i,2}]^T$ and (i, j, k) is cyclically permuted.

Then the local basis functions β_i, β_j and β_k are given by

$$\begin{bmatrix} \beta_i \\ \beta_j \\ \beta_k \end{bmatrix} = \frac{1}{2\,\mathrm{meas}(T)} \begin{bmatrix} g_i & a_i & b_i \\ g_j & a_j & b_j \\ g_k & a_k & b_k \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}, \qquad (5)$$

where meas(T) is the area of the triangle T and $x = (x_1, x_2) \in T$. Consequently,

$$\frac{\partial \beta_\ell}{\partial x_1} = \frac{a_\ell}{2\ \mathrm{meas}(T)}, \qquad \frac{\partial \beta_\ell}{\partial x_2} = \frac{b_\ell}{2\ \mathrm{meas}(T)}, \quad \ell = i, j, k. \qquad (6)$$

We consider the computation of the left-hand side of (2) on the triangle $m_a m_g m_c$,

$$q_i = -\int_{m_a m_g} (\mathbf{K}\,\nabla P_h) \cdot \hat{\mathbf{n}}\,ds - \int_{m_g m_c} (\mathbf{K}\,\nabla P_h) \cdot \hat{\mathbf{n}}\,ds\,. \qquad (7)$$

On the edge $m_a m_g$,

$$\hat{\mathbf{n}} = \frac{[m_{g,2} - m_{a,2},\ m_{a,1} - m_{g,1}]^T}{|\,m_a m_g\,|}$$

and, on the edge $m_g m_c$,

$$\hat{\mathbf{n}} = \frac{[m_{c,2} - m_{g,2},\ m_{g,1} - m_{c,1}]^T}{|\,m_g m_c\,|},$$

where $|\,m_a m_g\,|$ and $|\,m_g m_c\,|$ denote the lengths of edges $m_a m_g$ and $m_g m_c$, respectively. Consequently, if **K** is constant on the triangle T, it follows from (3), (6)-(7), the definition of a_i and b_i, and simple algebraic calculations that

$$q_i = \mathrm{meas}(T) \sum_{\ell=i}^{k} (\mathbf{K}\,\nabla \beta_\ell) \cdot (P_\ell\,\nabla \beta_i), \qquad (8)$$

where P_ℓ is the pressure at the vertices of triangles.

Equation in (8) can be recast in the finite difference form, by using (4), as

$$q_i = -T_{ij}(P_j - P_i) - T_{ik}(P_k - P_i), \qquad (9)$$

where the **transmissibility coefficients** T_{ij} and T_{ik} are

$$T_{ij} = -\mathrm{meas}(T)(\mathbf{K}\nabla\beta_j)\cdot\nabla\beta_i, \quad T_{ik} = -\mathrm{meas}(T)(\mathbf{K}\nabla\beta_k)\cdot\nabla\beta_i.$$

We now consider the assembly of the global transmissibility matrix. Each connection between any two adjacent nodes m_i and m_j includes the contributions from two triangles $T^{(1)}$ and $T^{(2)}$ that share the common edge with endpoints m_i and m_j. The transmissibility between m_i and m_j, where at least one of them is not on the external boundary, is

$$T_{ij} = -\sum_{\ell=1}^{2}\left[\mathrm{meas}(T)(\mathbf{K}\nabla\beta_j)\cdot\nabla\beta_i\right]\Big|_{T^{(\ell)}}. \tag{10}$$

Applying (2) and (9), we obtain the linear system on the control volume V_i in terms of pressure values at the vertices of triangles as

$$-\sum_{j\in N_i} T_{ij}(P_j - P_i) = F_i, \tag{11}$$

where N_i is the set of all neighbouring nodes of m_i and $F_i = \int_{V_i} f(x)\,dx$.

If ∂V_i contains part of the Neumann boundary, then the flux on that part must be given.If it contains part of the Dirichlet boundary, then the pressure on the corresponding part must be given. The third boundary condition can be also incorporated.

1.2 Positive transmissibilities

The transmissibility coefficient T_{ij} defined in (10) must be positive. Positive transmissibilities always yield a direction of the discrete flux in the physical direction. Negative transmissibilities are not physically meaningful and generate unsatisfactory solutions.

For simplicity, consider a homogeneous anisotropic medium with $\mathbf{K} = \mathrm{diag}(a_{11}, a_{22})$,where a_{11} and a_{22} are positive constants. In this case, using (6) and (10), T_{ij} restricted to each triangle T is

$$T_{ij} = -\frac{a_{11}\,a_j\,a_i + a_{22}\,b_j\,b_i}{4\,\mathrm{meas}(T)}.$$

Let us introduce a coordinate transform:

$$x'_1 = \frac{x_1}{\sqrt{a_{11}}}, \qquad x'_2 = \frac{x_2}{\sqrt{a_{22}}}.$$

Under this transform, the area of the transformed triangle T' is

$$\text{meas}(T') = \frac{\text{meas}(T)}{\sqrt{a_{11}\, a_{22}}}.$$

Consequently, T_{ij} becomes

$$T_{ij} = \sqrt{a_{11}\, a_{22}}\, \frac{|m_{k'}m_{j'}|\,|m_{k'}m_{i'}|\cos\theta_{k'}}{4\,\text{meas}(T')} = \sqrt{a_{11}a_{22}}\, \frac{\cot\theta_{k'}}{2},$$

where $\theta_{k'}$ is the angle of the triangle T' at node $m_{k'}$ in the transformed plane. Because each global transmissibility consists of the contributions from two adjacent triangles, the global T_{ij} between nodes m_i and m_j is

$$T_{ij} = \sqrt{a_{11}a_{22}}\, \frac{\cot\theta_{k'_1} + \cot\theta_{k'_2}}{2}, \tag{12}$$

where $\theta_{k'_1}$ and $\theta_{k'_2}$ are the opposite angles of the two triangles. Thus the requirement $T_{ij} > 0$ is equivalent to

$$\theta_{k'_1} + \theta_{k'_2} < \pi. \tag{13}$$

For an edge on the external boundary, the requirement for the angle opposite this edge is

$$\theta_{k'} < \frac{\pi}{2}. \tag{14}$$

We note that all these angles are measured in the (x'_1, x'_2)-coordinate plane.

1.3 Upstream weighted CVFEs

The basic idea of upstream weighting is to choose the value of a property coefficient according to the upstream direction of a flux. The same idea has been used in the upwind finite difference methods. In this section, we consider two upstream weighting strategies for (11).

(i) Potential-based upstream weighting schemes

Suppose that (1) is of the form

$$-\nabla \cdot (\lambda \mathbf{K} \nabla P) = f \qquad \text{in } \Omega, \tag{15}$$

where **K** and λ can be a permeability tensor and a mobility coefficient, respectively. For this problem, a CVFE method analogous to (11) can be derived. If **K** is a scalar a and is different on the two sides of an edge, across that edge it should be approximated by the **harmonic average**

$$a_{har}(x) = \frac{1}{\frac{1}{2}\left(\frac{1}{a^{+}(x)} + \frac{1}{a^{-}(x)}\right)} = \frac{2a^{+}(x)\, a^{-}(x)}{a^{+}(x) + a^{-}(x)},$$

where a^{+} and a^{-} indicate the respective values from the two sides. The reason for using a harmonic average is that for an inactive node (i.e., the node where a = 0), this average gives the correct value, in contrast with the arithmetic average. If **K** is a tensor, this harmonic average is used for each component of **K**, and the result is denoted by $\mathbf{K}_{har}$. For the mobility coefficient λ, in practice, upstream weighting must be used to maintain stability for the CVFE methods. As a result of these two observations, the transmissibility between nodes m_i and m_j restricted to each triangle T becomes

$$T_{ij} = -\text{meas}(T)\, \tilde{\lambda}_{ij}\, a_{har} \nabla \beta_i \cdot \nabla \beta_j, \tag{16}$$

where the potential-based upstream weighting scheme is defined by

$$\tilde{\lambda}_{ij} = \begin{cases} \lambda(m_i) & \text{if } P_i > P_j, \\ \lambda(m_j) & \text{if } P_i < P_j. \end{cases} \tag{17}$$

In fact, it is a pressure-based approach in the current context. The name potential-based is due to the fact that potentials are usually used in place of P in reservoir simulation.

This potential-based upstream weighting scheme is easy to implement. However, it violates the important flux continuity property across the interfaces between control volumes. To see this, consider the case $\mathbf{K} = \mathrm{diag}(a_{11}, a_{22})$, where $\mathbf{K}$ is a constant diagonal tensor on the triangle T. Applying (6) and (16), the flux on edge $m_a m_g$ is

$$q_{i,m_a m_g} = -\tilde{\lambda}_{ij}\left(a_{11}(m_{g,2} - m_{a,2})\frac{\partial \beta_j}{\partial x_1} + a_{22}(m_{a,1} - m_{g,1})\frac{\partial \beta_j}{\partial x_2} \right)(P_j - P_i)$$
$$-\tilde{\lambda}_{ik}\left(a_{11}(m_{g,2} - m_{a,2})\frac{\partial \beta_k}{\partial x_1} + a_{22}(m_{a,1} - m_{g,1})\frac{\partial \beta_k}{\partial x_2} \right)(P_k - P_i),$$

and on edge $m_g m_c$

$$q_{i,m_g m_c} = -\tilde{\lambda}_{ij}\left(a_{11}(m_{c,2} - m_{g,2})\frac{\partial \beta_j}{\partial x_1} + a_{22}(m_{g,1} - m_{c,1})\frac{\partial \beta_j}{\partial x_2} \right)(P_j - P_i)$$
$$-\tilde{\lambda}_{ik}\left(a_{11}(m_{c,2} - m_{g,2})\frac{\partial \beta_k}{\partial x_1} + a_{22}(m_{g,1} - m_{c,1})\frac{\partial \beta_k}{\partial x_2} \right)(P_k - P_i).$$

Similarly, the fluxes on edges $m_b m_g$ and $m_g m_a$ at node m_j are, respectively,

$$q_{j,m_b m_g} = -\tilde{\lambda}_{jk}\left(a_{11}(m_{g,2} - m_{b,2})\frac{\partial \beta_k}{\partial x_1} + a_{22}(m_{b,1} - m_{g,1})\frac{\partial \beta_k}{\partial x_2} \right)(P_k - P_j)$$
$$-\tilde{\lambda}_{ji}\left(a_{11}(m_{g,2} - m_{b,2})\frac{\partial \beta_i}{\partial x_1} + a_{22}(m_{b,1} - m_{g,1})\frac{\partial \beta_i}{\partial x_2} \right)(P_i - P_j)$$

and

$$q_{j,m_g m_a} = -\tilde{\lambda}_{jk}\left(a_{11}(m_{a,2} - m_{g,2})\frac{\partial \beta_k}{\partial x_1} + a_{22}(m_{g,1} - m_{a,1})\frac{\partial \beta_k}{\partial x_2} \right)(P_k - P_j)$$
$$-\tilde{\lambda}_{ji}\left(a_{11}(m_{a,2} - m_{g,2})\frac{\partial \beta_i}{\partial x_1} + a_{22}(m_{g,1} - m_{a,1})\frac{\partial \beta_i}{\partial x_2} \right)(P_i - P_j).$$

And the fluxes on edges $m_c m_g$ and $m_g m_b$ at node m_k are, respectively,

$$q_{k,m_c m_g} = -\tilde{\lambda}_{ki}\left(a_{11}(m_{g,2} - m_{c,2})\frac{\partial \beta_i}{\partial x_1} + a_{22}(m_{c,1} - m_{g,1})\frac{\partial \beta_i}{\partial x_2} \right)(P_i - P_k)$$
$$-\tilde{\lambda}_{kj}\left(a_{11}(m_{g,2} - m_{c,2})\frac{\partial \beta_j}{\partial x_1} + a_{22}(m_{c,1} - m_{g,1})\frac{\partial \beta_j}{\partial x_2} \right)(P_j - P_k)$$

and

$$q_{k,m_g m_b} = -\tilde{\lambda}_{ki}\left(a_{11}(m_{b,2} - m_{g,2})\frac{\partial\beta_i}{\partial x_1} + a_{22}(m_{g,1} - m_{b,1})\frac{\partial\beta_i}{\partial x_2} \right)(P_i - P_k)$$
$$-\tilde{\lambda}_{kj}\left(a_{11}(m_{b,2} - m_{g,2})\frac{\partial\beta_j}{\partial x_1} + a_{22}(m_{g,1} - m_{b,1})\frac{\partial\beta_j}{\partial x_2} \right)(P_j - P_k).$$

For the flux to be continuous across edge $m_a m_g$, it is required that $q_{i,m_a m_g} + q_{j,m_g m_a} = 0$; i.e.,

$$-\tilde{\lambda}_{ij}\left(a_{11}(m_{g,2} - m_{a,2})\frac{\partial\beta_j}{\partial x_1} + a_{22}(m_{a,1} - m_{g,1})\frac{\partial\beta_j}{\partial x_2} \right)(P_j - P_i)$$
$$-\tilde{\lambda}_{ik}\left(a_{11}(m_{g,2} - m_{a,2})\frac{\partial\beta_k}{\partial x_1} + a_{22}(m_{a,1} - m_{g,1})\frac{\partial\beta_k}{\partial x_2} \right)(P_k - P_i)$$
$$-\tilde{\lambda}_{jk}\left(a_{11}(m_{a,2} - m_{g,2})\frac{\partial\beta_k}{\partial x_1} + a_{22}(m_{g,1} - m_{a,1})\frac{\partial\beta_k}{\partial x_2} \right)(P_k - P_j)$$
$$-\tilde{\lambda}_{ji}\left(a_{11}(m_{a,2} - m_{g,2})\frac{\partial\beta_i}{\partial x_1} + a_{22}(m_{g,1} - m_{a,1})\frac{\partial\beta_i}{\partial x_2} \right)(P_i - P_j) = 0\ .$$

Because it must be satisfied for all choices of **K**, this equation reduces to

$$a_{11}(m_{a,2} - m_{g,2})\left[\tilde{\lambda}_{ij}\frac{\partial\beta_j}{\partial x_1}(P_j - P_i) + \tilde{\lambda}_{ji}\frac{\partial\beta_i}{\partial x_1}(P_j - P_i) \right.$$
$$\left. + \tilde{\lambda}_{ik}\frac{\partial\beta_k}{\partial x_1}(P_k - P_i) + \tilde{\lambda}_{jk}\frac{\partial\beta_k}{\partial x_1}(P_j - P_k) \right] = 0$$

and

$$a_{22}(m_{g,1} - m_{a,1})\left[\tilde{\lambda}_{ij}\frac{\partial\beta_j}{\partial x_2}(P_j - P_i) + \tilde{\lambda}_{ji}\frac{\partial\beta_i}{\partial x_2}(P_j - P_i) \right.$$
$$\left. + \tilde{\lambda}_{ik}\frac{\partial\beta_k}{\partial x_2}(P_k - P_i) + \tilde{\lambda}_{jk}\frac{\partial\beta_k}{\partial x_2}(P_j - P_k) \right] = 0.$$

For these two equations to hold simultaneously for any type of triangle, the only possibility is

$$P_k \geq P_i = P_j.$$

In the same manner, we can prove

$$P_i \geq P_j = P_k \text{ and } P_j \geq P_i = P_k .$$

Hence, for the flux to be continuous across the edges, $P_i = P_j = P_k$. That is, the flux is continuous across the edges if and only if the approximate solution P_h has the same value at all vertices, which is generally not true. Therefore, in general, the potential-based upstream weighted CVFE method generates a discontinuous flux across the edges. On the other hand, the above argument leads to another upstream weighting strategy.

(ii) Flux-based upstream weighting schemes

For the flux-based approach, the upstream direction is determined by the sign of a flux. It follows from (7) and (16) that the flux on edge $m_a m_g$ at node m_i is

$$q_{i,m_a m_g} = -\sum_{\ell=i}^{k} \tilde{\lambda} \mathbf{K}_{har} \nabla\lambda_\ell \cdot [m_{g,2} - m_{a,2}, m_{a,1} - m_{g,1}]^T P_\ell,$$

and, at node m_j,

$$q_{j,m_g m_a} = -\sum_{\ell=i}^{k} \tilde{\lambda} \mathbf{K}_{har} \nabla\lambda_\ell \cdot [m_{a,2} - m_{g,2}, m_{g,1} - m_{a,1}]^T P_\ell,$$

where the upstream weighting is now defined by

$$\tilde{\lambda} = \begin{cases} \lambda(m_i) & \text{if } q_{i,m_a m_g} > 0, \\ \lambda(m_j) & \text{if } q_{i,m_a m_g} < 0. \end{cases} \tag{18}$$

From this definition it follows that

$$q_{i,m_a m_g} + q_{j,m_g m_a} = 0. \tag{19}$$

The fluxes on other edges can be defined in the same fashion. It is evident from (19) that the flux-based upstream weighted CVFE method has a continuous flux across the edges.

2 Mixed Finite Element Methods

2.1 Finite element spaces on tetrahedra

Let T_h be a partition of $\Omega \subset \mathbb{R}^3$ into tetrahedra such that adjacent elements completely share their common face. In three dimensions, P_k is now the space of polynomials of degree k in three variables x_1, x_2 and x_3.

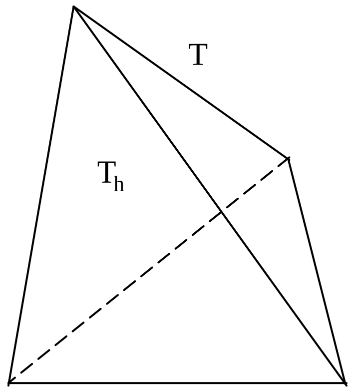

Figure 2: Tetrahedron T_h

(i) Raviart-Thomas-Nédélec (RTN) spaces on tetrahedra

These spaces are the three-dimensional analogues of the Raviart-Thomas(RT) spaces on triangles and they are defined for each k≥ 0 by

$$V_h(T) = (P_k(T))^3 \oplus ((x_1, x_2, x_3)P_k(T)), \qquad W_h(T) = P_k(T),$$

where $(x_1, x_2, x_3)P_k(T) = (x_1 P_k(T), x_2 P_k(T), x_3 P_k(T))$. As in two dimensions, for k = 0, V_h is

$$V_h(T) = \{v \mid \ v = (a_T + d_T x_1, b_T + d_T x_2, c_T + d_T x_3), \quad a_T, b_T, c_T, d_T \in \mathbb{R}\},$$

and its dimension is four. The degrees of freedom are the values of normal components of functions at the centroid of each face T.

In general, for k≥ 0, the dimensions of $V_h(T)$ and $W_h(T)$ are

$$\dim(V_h(T)) = \frac{(k+1)(k+2)(k+4)}{2},$$

$$\dim(W_h(T)) = \frac{(k+1)(k+2)(k+3)}{6}.$$

(ii) Brezzi-Douglas-Durán-Fortin(BDDF) spaces on tetrahedra

The BDDF spaces are an extension of the Brezzi-Douglas-Marini(BDM) spaces on triangles to tetrahedra, and they are given for each $k \geq 1$ by

$$V_h(T) = \left(P_k(T)\right)^3, \qquad W_h(T) = P_{k-1}(T).$$

The dimensions of $V_h(T)$ and $W_h(T)$ are

$$\dim\left(V_h(T)\right) = \frac{(k+1)(k+2)(k+3)}{2},$$
$$\dim\left(W_h(T)\right) = \frac{k(k+1)(k+2)}{6}.$$

3 Characteristic Finite Element Methods

In this section, we consider an application of finite element methods to the reaction-diffusion-advection problem:

$$\frac{\partial(\phi P)}{\partial t} + \nabla \cdot (P\mathbf{b} - \mathbf{A}\nabla P) + RP = f, \qquad (20)$$

for the unknown solution P, where ϕ, $\mathbf{b}$, $\mathbf{A}$,R and f are given functions. We note that (20) involves advection $\mathbf{b}$, diffusion $\mathbf{A}$, and reaction R. Many equations arise in this form, e.g., saturation and concentration equations for multiphase, multicomponent flows in porous media.

The modified method of characteristics (MMOC) was independently developed by Douglas and Russel (1982) and Pironneau (1982). It is based on a nondivergence form of (20). In the engineering literature the name Eulerian- Lagrangian method is often used.

3.1 A one-dimensional model problem

For the purpose of introduction, we consider a one-dimensional model problem on the whole real line:

$$\phi(x)\frac{\partial P}{\partial t} + b(x)\frac{\partial P}{\partial x} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial P}{\partial x}\right) + R(x,t)P = f(x,t), \quad x \in \mathbb{R},\ t > 0, \qquad (21)$$

$$P(x,0) = P_0(x), \quad x \in \mathbb{R}. \qquad (22)$$

Set

$$\psi(x) = \left[(\phi(x))^2 + (b(x))^2 \right]^{\frac{1}{2}}.$$

Assume that

$$\phi(x) > 0, \quad x \in \mathbb{R},$$

so that $\psi(x) > 0$, $x \in \mathbb{R}$. Let the characteristic direction, $\phi \dfrac{\partial P}{\partial t} + b \dfrac{\partial P}{\partial x}$, associated with the hyperbolic part of (21) be denoted by y, so that

$$\frac{\partial}{\partial y} = \frac{\phi(x)}{\psi(x)} \frac{\partial}{\partial t} + \frac{b(x)}{\psi(x)} \frac{\partial}{\partial x}.$$

Then (21) can be rewritten as

$$\psi(x) \frac{\partial P}{\partial y} - \frac{\partial}{\partial x}\left(a(x,t) \frac{\partial P}{\partial x} \right) + R(x,t)\, P = f(x,t), \quad x \in \mathbb{R}, \ t > 0. \qquad (23)$$

We assume that the coefficients a, b, R and ϕ are bounded and satisfy

$$\left| \frac{b(x)}{\phi(x)} \right| + \left| \frac{d}{dx} \frac{b(x)}{\phi(x)} \right| \le C, \qquad x \in \mathbb{R},$$

where C is a positive constant. We introduce the linear space

$$V = H^1(\mathbb{R}).$$

We recall the scalar product

$$\langle v, w \rangle = \int_{\mathbb{R}} v(x)\, w(x)\, dx$$

in $L^2(\mathbb{R})$. Now, multiplying (23) by any $v \in V$ and applying integration by parts in space, (23) can be written in the equivalent variational form

$$\left\langle \psi \frac{\partial P}{\partial y}, v \right\rangle + \left\langle a \frac{\partial P}{\partial x}, \frac{dv}{dx} \right\rangle + \langle RP, v \rangle = \langle f, v \rangle, \qquad v \in V, t > 0, \qquad (24)$$

Let $0 = t^0 < t^1 < \cdots < t^n < \cdots$ be a partition in time, with $\Delta t^n = t^n - t^{n-1}$. For a generic function v of time, set $v^n = v(\cdot, t^n)$. The characteristic derivative is approximated in the following way. Let

$$\hat{x}^n = x - \frac{\Delta t^n}{\phi(x)} b(x). \qquad (25)$$

We note that, at $t = t^n$,

$$\psi \frac{\partial P}{\partial y} \approx \psi(x) \frac{P(x, t^n) - P(\hat{x}^n, t^{n-1})}{\left[(x - \hat{x}^n)^2 + (\Delta t^n)^2 \right]^{\frac{1}{2}}}$$

$$= \phi(x) \frac{P(x, t^n) - P(\hat{x}^n, t^{n-1})}{\Delta t^n}. \qquad (26)$$

That is, a backtracking algorithm is used to approximate the characteristic derivative. $\hat{x}^n$ is the foot at level t^{n-1} of the characteristic corresponding to x at the head, at level t^n.

Let V_h be a finite element subspace of $V \cap W^{1,\infty}(\mathbb{R})$. Because we are considering the whole line, V_h is necessarily infinite-dimensional. In practice, we can assume that the support of P_0 is compact, the portion of the line on which we need to know P is bounded, and P is very small outside that set. Then V_h can be taken to be finite-dimensional.

The MMOC for (21) is defined as follows:

For n = 1, 2, …, find $P_h^n \in V_h$ such that

$$\left\langle \phi \frac{P_h^n - P_h^{n-1}}{\Delta t^n}, v \right\rangle + \left\langle a^n \frac{dP_h^n}{dx}, \frac{dv}{dx} \right\rangle + \left\langle R^n P_h^n, v \right\rangle = \left\langle f^n, v \right\rangle \text{ for all } v \in V_h, \quad (27)$$

where

$$P_h^{n-1} = P_h(\hat{x}^n, t^{n-1}) = P_h \left\langle x - \frac{\Delta t^n}{\phi(x)} b(x), t^{n-1} \right\rangle. \qquad (28)$$

The initial approximation P_h^0 can be defined as the interpolant of P_0 in V_h.

Equation in (27) determines $\{P_h^n\}$ uniquely in terms of the data P_0 and f. This can be seen as follows. Since (27) is a finite-dimensional system, it suffices to show uniqueness of the solution. Let $f = P_0 = 0$, and take $v = P_h^n$ in (27) to see that

$$\left\langle \phi \frac{P_h^n - P_h^{n-1}}{\Delta t^n}, P_h^n \right\rangle + \left\langle a^n \frac{dP_h^n}{dx}, \frac{dP_h^n}{dx} \right\rangle + \left\langle R^n P_h^n, P_h^n \right\rangle = 0,$$

with an assumption that $P_h^{n-1} = 0$, this equation implies $P_h^n = 0$.

It is obvious that the linear system arising from (27) is symmetric positive definite, even in the presence of the advection term.

We end with a remark on a convergence result for (27). Let $V_h \subset V$ be a finite element space with the following approximation property:

$$\inf_{v_h \in V_h} \left\{ \| v - v_h \|_{L^2(\Omega)} + h \| v - v_h \|_{H^1(\Omega)} \right\} \le C h^{k+1} | v |_{H^{k+1}(\Omega)}, \qquad (29)$$

where the constant C > 0 is independent of h, and k> 0 is an integer. Then, under appropriate assumptions on the smoothness of the solution P and a suitable choice of P_h^0 it can be shown that

$$\max_{1 \le n \le N} \left\{ \| P^n - P_h^n \|_{L^2(\Omega)} + h \| P^n - P_h^n \|_{H^1(\Omega)} \right\} \le C(P)(h^{k+1} + \Delta t), \qquad (30)$$

where N is an integer such that $t^N = T < \infty$ and [0,T] is the time interval of interest.

3.2 Periodic boundary conditions

In the previous subsection, (21) was considered on the whole line. For an interval (0,1), the MMOC has a difficulty in handling general boundary conditions. In this case, it is normally developed for periodic boundary conditions

$$P(0,t) = P(1,t), \qquad \frac{\partial P}{\partial x}(0,t) = \frac{\partial P}{\partial x}(1,t). \qquad (31)$$

In the periodic case, assume that all functions in (21) are spatially (0,1)-periodic. Accordingly, the linear space V is modified to

$$V = \{ v \in H^1(0,1) \mid \ v \text{ is } (0,1)\text{-periodic} \}.$$

With this modification, the developments in (24) and (27) remain unchanged.

3.3 Extension to two- and three-dimensional problems

We now extend the MMOC to (20) defined on a higher-dimensional domain. Let $\Omega \subset \mathbb{R}^d$ (d = 2 or 3) be a rectangle (respectively, a rectangular parallelepiped), and assume that all functions in (20) are spatially Ω-periodic. We write (20) in nondivergence form:

$$\phi(x)\frac{\partial P}{\partial t} + \mathbf{b}(x,t)\cdot\nabla P - \nabla\cdot(\mathbf{A}(x,t)\nabla P) + R(x,t)\,P = f(x,t), \;\; x\in\Omega,\; t>0, \quad (32)$$

$$P(x,0) = P_0(x), \qquad x\in\Omega. \qquad (33)$$

Set

$$\psi(x,t) = \left[(\phi(x))^2 + |\mathbf{b}(x,t)|^2\right]^{\frac{1}{2}}.$$

Assume that

$$\phi(x) > 0, \quad x \in\Omega.$$

Now, if the characteristic direction, $\phi\frac{\partial P}{\partial t} + \mathbf{b}\cdot\nabla P$, corresponding to the hyperbolic part of (32) is y, then

$$\frac{\partial}{\partial y} = \frac{\phi(x)}{\psi(x,t)}\frac{\partial}{\partial t} + \frac{1}{\psi(x,t)}\mathbf{b}(x,t)\cdot\nabla.$$

With this definition, (32) becomes

$$\psi(x,t)\frac{\partial P}{\partial y} - \nabla\cdot\left(\mathbf{A}(x,t)\nabla P\right) + R(x,t)\,P = f(x,t), \quad x\in\Omega,\; t>0, \;(34)$$

We define the linear space

$$V = \{v \in H^1(\Omega) \,|\; v \text{ is } \Omega\text{-periodic}\}.$$

Recall the notations

$$\langle v, w\rangle_S \;=\; \int_S v(x)\,w(x)\,dx\,,$$

$$\langle \nabla v, \nabla w\rangle_S \;=\; \int_S \nabla v(x)\cdot\nabla w(x)\,dx\,.$$

If S = Ω, we omit it in this notation. Now, applying Green’s formula in space and the periodic boundary conditions, (34) can be written in the equivalent variational form:

$$\left\langle \psi\frac{\partial P}{\partial y}, v\right\rangle + \langle \mathbf{A}\nabla P, \nabla v\rangle + \langle RP, v\rangle = \langle f, v\rangle, \;\; \text{for all } v\in V,\; t>0. \quad (35)$$

The characteristic is approximated by

$$\hat{\mathbf{x}}^n = \mathbf{x} - \frac{\Delta t^n}{\phi(x)} \mathbf{b}(x, t^n), . \tag{36}$$

where $x = (x_1, \ldots, x_d)$ and $\mathbf{x} = [x_1, \ldots, x_d]^T$. Furthermore, we see that, at $t = t^n$,

$$\begin{aligned} \psi \frac{\partial P}{\partial y} &\approx \psi(x, t^n) \frac{P(x, t^n) - P(\hat{x}^n, t^{n-1})}{\left[|\mathbf{x} - \hat{\mathbf{x}}^n|^2 + (\Delta t^n)^2 \right]^{\frac{1}{2}}} \\ &= \phi(x) \frac{P(x, t^n) - P(\hat{x}^n, t^{n-1})}{\Delta t^n}. \end{aligned} \tag{37}$$

A backtracking algorithm similar to that employed in one dimension is used to approximate the characteristic derivative.

Let $V_h \subset V$ be a finite element space associated with a partition T_h of Ω. The MMOC for (32) is given as follows:

For n = 1, 2, …, find $P_h^n \in V_h$ such that

$$\left\langle \phi \frac{P_h^n - P_h^{n-1}}{\Delta t^n}, v \right\rangle + \left\langle \mathbf{A}^n \nabla P_h^n, \nabla v \right\rangle + \left\langle R^n P_h^n, v \right\rangle = \left\langle f^n, v \right\rangle \text{ for all } v \in V_h, \tag{38}$$

where

$$P_h^{n-1} = P_h(\hat{x}^n, t^{n-1}). \tag{39}$$

Existence and uniqueness of a solution for **A** and R can be shown, and the error estimate (30) under appropriate assumptions on P also holds for (38) provided that an approximation property similar to (24) holds for V_h in the multiple dimensions.

4 Finite Element Methods for One-Phase Flows

We now turn to the three-dimensional one-phase flow equation. We write it in a more general form:

$$c(P)\frac{\partial P}{\partial t} - \nabla \cdot \left(\mathbf{K}(P) \nabla P \right) = f(P) \qquad \text{in } \Omega \times (0,T), \tag{40}$$

$$(\mathbf{K}(P)\nabla P) \cdot \hat{\mathbf{n}} = 0 \qquad \text{on } \Gamma \times (0,T), \tag{41}$$

$$P(\cdot, 0) = P_0 \qquad \text{in } \Omega, \tag{42}$$

where P is the pressure, **K** is the absolute permeability tensor, $\hat{\mathbf{n}}$ is the outward unit normal to the boundary Γ of Ω, the function P_0 is given and (0,T) is the time interval of interest.

We assume that (40)-(42) admits a unique solution. Furthermore, we assume that the coefficients c, **K** and f are Lipschitz continuousin P.

With $V = H^1(\Omega)$, (40)-(41) can be written in the variational form:

Find P : (0, T)$\rightarrow$ V such that

$$t \mapsto P(\cdot, t)$$

$$\left\langle c(P)\frac{\partial P}{\partial t}, v\right\rangle + \left\langle \mathbf{K}(P)\nabla P, \nabla v\right\rangle = \langle f(P), v\rangle \text{for all } v \in V,\ t \in (0,T). \quad (43)$$

Let V_h be a finite element subspace of V. The finite element version of (43) and (42) is as follows:

Find $P_h : (0,T) \rightarrow V_h$ such that

$$t \mapsto P_h(\cdot, t)$$

$$\left\langle c(P_h)\frac{\partial P_h}{\partial t}, v_h\right\rangle + \left\langle \mathbf{K}(P_h)\ \nabla P_h, \nabla v_h\right\rangle = \langle f(P_h), v_h\rangle \quad \text{for all } v_h \in V_h, \quad (44)$$

$$\langle P_h(\cdot, 0), v_h\rangle = \langle P_0, v_h\rangle \qquad \text{for all } v_h \in V_h. \quad (45)$$

After the introduction of basis functions in V_h, (44)-(45) can be stated in matrix form as

$$\mathbf{C}(\mathbf{P})\ \frac{d\mathbf{P}}{dt} + \mathbf{K}(\mathbf{P})\ \mathbf{P} = \mathbf{f}\,(\mathbf{P}), \qquad t \in (0, T), \quad (46)$$

$$\mathbf{B}\mathbf{P}(0) = \mathbf{P}_0. \quad (47)$$

Under the assumption that the coefficient c(P) is bounded below by a positive constant, this nonlinear system of ordinary differential equations possesses a unique solution. In fact, because of the assumption on c, **A** and f, the solution **P**(t) exists for all t.

4.1 Linearization approaches

Let $0 = t^0 < t^1 < t^2 < \cdots < t^N = T$ be a partition of (0,T), and set $\Delta t^n = t^n - t^{n-1}$, or n = 1, 2, …, N. The nonlinear system in (46)-(47) can be linearized by allowing the nonlinearities to lag one time step behind. Thus the modified backward Euler method for (40)-(42) takes the form:

Find $P_h^n \in V_h$, n = 1, 2, …, N, such that

$$\left\langle c(P_h^{n-1})\,\frac{P_h^n - P_h^{n-1}}{\Delta t^n}, v_h \right\rangle + \left\langle \mathbf{K}(P_h^{n-1})\nabla P_h^n, \nabla v_h \right\rangle = \left\langle f(P_h^{n-1}), v_h \right\rangle$$

$$\text{for all } v_h \in V_h, \qquad (48)$$

$$\left\langle P_h^0, v_h \right\rangle = \left\langle P_0, v_h \right\rangle \quad \text{for all } v_h \in V_h. \qquad (49)$$

In matrix form it is given by

$$\mathbf{C}(\mathbf{P}^{n-1})\,\frac{\mathbf{P}^n - \mathbf{P}^{n-1}}{\Delta t} + \mathbf{K}(\mathbf{P}^{n-1})\mathbf{P}^n = \mathbf{f}(\mathbf{P}^{n-1}), \qquad (50)$$

$$\mathbf{BP}(0) = \mathbf{P}_0. \qquad (51)$$

We note that (50)-(51) is a system of linear equations in $\mathbf{P}^n$, which can be solved by using iterative algorithms.We may use the Crank-Nicholson discretization method in (48). However, the linearization decreases the order of the time discretization error to $O(\Delta t)$. This is true for any higher order time discretization method with the present linearization technique. This drawback can be overcome by using extrapolation techniques in the linearization of the coefficients c, **K** and f. Combined with an appropriate extrapolation, the Crank-Nicholson method can be shown to produce an error of order $O((\Delta t)^2)$ in time.

4.2 Implicit time approximations

We now consider a fully implicit time approximation method for (40)-(42):

Find $P_h^n \in V_h$, n = 1, 2, …, N, such that

$$\left\langle c(P_h^n)\,\frac{P_h^n - P_h^{n-1}}{\Delta t^n}, v_h \right\rangle + \left\langle \mathbf{K}(P_h^n)\,\nabla P_h^n, \nabla v_h \right\rangle = \left\langle f(P_h^n), v_h \right\rangle$$

$$\text{for all } v_h \in V_h, \qquad (52)$$

with (49). Its matrix form is

$$\mathbf{C}(\mathbf{P}^n)\frac{\mathbf{P}^n-\mathbf{P}^{n-1}}{\Delta t^n}+\mathbf{K}(\mathbf{P}^n)\ \mathbf{P}^n \ = \ \mathbf{f}(\mathbf{P}^n). \tag{53}$$

Now, (53) is a system of nonlinear equations in $\mathbf{P}^n$, which must be solved at each time step via an iteration method.

4.3 Explicit time approximations

We conclude with a remark about the application of a forward explicit time approximation method to (40)-(42):

Find $P_h^n \in V_h$, $n = 1, 2, \ldots, N$, such that

$$\left\langle c(P_h^n)\frac{P_h^n-P_h^{n-1}}{\Delta t^n}, v_h\right\rangle+\left\langle \mathbf{K}(P_h^{n-1})\ \nabla P_h^{n-1}, \nabla v_h\right\rangle=\left\langle f(P_h^{n-1}), v_h\right\rangle$$

$$\text{for all } v_h \in V_h, \tag{54}$$

with (49). In matrix form it is written as

$$\mathbf{C}(\mathbf{P}^n)\ \frac{\mathbf{P}^n-\mathbf{P}^{n-1}}{\Delta t}+\mathbf{K}(\mathbf{P}^{n-1})\mathbf{P}^{n-1}=\mathbf{f}(\mathbf{P}^{n-1}). \tag{55}$$

We note that the nonlinearity is only in **C**.

For the explicit method in (54) to be stable, a stability condition of the following type must be satisfied:

$$\Delta t^n \le C h^2, \qquad n = 1, 2, \ldots, N, \tag{56}$$

where C now depends on c and **K**. Unfortunately, this condition on the time steps is very restrictive for long-time integration.

5 Finite Element Methods for Two-Phase Flows

The various discretization methods are now applied to the solution of the differential equations

$$\left.\begin{aligned}\frac{\partial}{\partial t}(\phi\rho_w S_w) &= -\nabla\cdot(\rho_w \mathbf{u}_w)+q_w,\\ \frac{\partial}{\partial t}(\phi\rho_o S_o) &= -\nabla\cdot(\rho_o \mathbf{u}_o)+q_o,\end{aligned}\right\} \tag{57}$$

where ϕ is the porosity of the porous medium; S_w, ρ_w, $\mathbf{u}_w$ and q_w are saturation, density, Darcy's velocity and mass flow rate for wetting phase; S_o, ρ_o, $\mathbf{u}_o$ and q_o are saturation, density, Darcy's velocity and mass flow rate for nonwetting phase;

$$\left.\begin{aligned} \mathbf{u}_w &= -\frac{m_w}{\mu_w}\mathbf{K}(\nabla P_w - \rho_w g \nabla z), \\ \mathbf{u}_0 &= -\frac{m_o}{\mu_o}\mathbf{K}(\nabla P_o - \rho_o g \nabla z), \end{aligned}\right\} \tag{58}$$

where **K** is the absolute permeability tensor of the porous medium; m_w, P_w and μ_w are the relative permeability, pressure and viscosity for wetting phase; m_o, P_o and μ_o are the relative permeability, pressure and viscosity for nonwetting phase; g is the magnitude of the gravitational acceleration; and z is the depth. Here

$$S_w + S_o = 1, \tag{59}$$

$$P_c(S_w) = P_o - P_w, \tag{60}$$

where P_c is the capillary pressure governing two-phase flows in a porous medium $\Omega \subset \mathbb{R}^3$.The standard finite element methods for one-phase flows can be extended to the present case. Here we discuss the application of the mixed, control volume and characteristic finite element methods to (57)-(60). The first two methods are good choices for the pressure equation.

5.1 Mixed finite element methods for two-phase flows

As an example, we present mixed finite element methods for the global formulation. Recall that the pressure equation consists of

$$\nabla \cdot \mathbf{u} = \tilde{q} \tag{61}$$

and

$$\mathbf{u} = -\mathbf{K}(\lambda(S)\nabla P - (\lambda_w \rho_w + \lambda_o \rho_o) g \nabla z), \tag{62}$$

where **u** is the total velocity, in this formulation. The model is completed by specifying boundary and initial conditions. For simplicity, a no-flow boundary conditions used for the pressure equation

$$\mathbf{u}(x,t) \cdot \hat{\mathbf{n}}(x,t) = 0, \qquad x \in \Gamma,\ t > 0. \tag{63}$$

It follows from (61) and (63) that compatibility of the fluids requires

$$\int_{\Omega} \tilde{q}\, dx = 0, \qquad t \ge 0 .$$

Set

$$\begin{aligned} H(\mathrm{div}, \Omega) &= \{v \in (H^1(\Omega))^3 \mid \mathrm{div}\, v = 0 \ \text{ in } \ \Omega\}, \\ \mathbf{V} &= \{\mathbf{v} \in H(\mathrm{div}, \Omega) \mid \ \mathbf{v} \cdot \hat{\mathbf{n}} = 0 \ \text{on} \ \Gamma\}, \\ W &= L^2(\Omega) . \end{aligned}$$

For simplicity, let Ω be a convex polygonal domain. For 0 < h < l, let T_h be a regular partition of Ω into elements, say, tetrahedra, with maximum mesh size h. Associated with the partition T_h, let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ represent the mixed finite element spaces. Now, the mixed method for (61)-(62) is as follows:

For $0 \le n \le N$, find $\mathbf{u}_h^n \in \mathbf{V}_h$ and $P_h^n \in W_h$ such that

$$\left\langle \nabla \cdot \mathbf{u}_h^n, w_h \right\rangle = \left\langle \tilde{q}(P_h^n, S_h^n), w_h \right\rangle \text{ for all } w_h \in W_h, \qquad (64)$$

$$\left\langle \mathbf{K}\, \lambda(S_h^n)^{-1} \mathbf{u}_h^n, v_h \right\rangle - \left\langle P_h^n, \nabla \cdot v_h \right\rangle = \left\langle \gamma(S_h^n), v_h \right\rangle \text{for all } v_h \in V_h, \qquad (65)$$

where S_h^n is an approximation to S^n and

$$\gamma(S) = \left(f_w(S)\, \rho_w + f_o(S)\, \rho_o \right) g \nabla z. \qquad (66)$$

We note that system in (64)-(65) is nonlinear.The various approaches developed in the preceding section for the standard finite element methods can be applied to it in the same fashion.

5.2 CVFE methods

Assume that a partition T_h of Ω consists of a set of control volumes V_i:

$$\overline{\Omega} = \bigcup_i \overline{V}_i, \qquad V_i \cap V_j = \emptyset, \ i \ne j.$$

On each V_i, integration of (61) over V_i and application of the divergence theorem gives

$$\int_{\partial V_i} \mathbf{u} \cdot \hat{\mathbf{n}}\, ds = \int_{V_i} \tilde{q}\, dx . \qquad (67)$$

Substituting (62) into this equation yields

$$-\int_{\partial V_i} \lambda(S)(\mathbf{K}\nabla P)\cdot\hat{\mathbf{n}}\,ds = \int_{V_i} \tilde{q}\,dx - \int_{\partial V_i} (\lambda_w(S)\,\rho_w + \lambda_o(S)\,\rho_o)\,g\,(\mathbf{K}\nabla z)\cdot\hat{\mathbf{n}}\,ds. \tag{68}$$

Let $V_h \subset H^1(\Omega)$ be a finite element space associated with the CVFE partition T_h. Then the CVFE method for the pressure equation reads:

For $0 \le n \le N$, find $p_h^n \in V_h$ such that

$$-\int_{\partial V_i} \lambda(S_h^n)\,(\mathbf{K}\nabla P_h^n)\cdot\hat{\mathbf{n}}\,ds = \int_{V_i} \tilde{q}(P_h^n, S_h^n)\,dx - \int_{\partial V_i} \left(\lambda_w(S_h^n)\,\rho_w + \lambda_o(S_h^n)\,\rho_o\right) g\,(\mathbf{K}\nabla z)\cdot\hat{\mathbf{n}}\,ds. \tag{69}$$

The upstream weighting techniques introduced in Subsection 1.3 can be applied to (69).

5.3 Characteristic finite element methods

As an example, we present the MMOC described in Section 3 for the saturation. Introduce

$$\tilde{q}_1(P,S) = \tilde{q}_w(P,S) - \tilde{q}(P,S)\,f_w(S) + \nabla\cdot\left(\mathbf{K}\,f_w(S)\,\lambda_o(S)(\rho_o - \rho_w)\,g\,\nabla z\right).$$

Using (61) and

$$\phi\frac{\partial S}{\partial t} + \nabla\cdot\left(\mathbf{K}\,f_w(S)\lambda_o(S)\left(\frac{dP_c}{dS}\nabla S + (\rho_o - \rho_w)\,g\,\nabla z\right) + f_w(S)\,\mathbf{u}\right) = \tilde{q}_w(P,S),$$

the saturation equation becomes

$$\phi\frac{\partial S}{\partial t} + \frac{df_w}{dS}\mathbf{u}\cdot\nabla S + \nabla\cdot\left(\mathbf{K}\,f_w(S)\,\lambda_o(S)\,\frac{dP_c}{dS}\,\nabla S\right) = q_1(P,S). \tag{70}$$

Let

$$\mathbf{b}(x,t) = \frac{df_w}{dS}\mathbf{u},$$

$$\psi(x,t) = \left[(\phi(x))^2 + |\mathbf{b}(x,t)|^2\right]^{\frac{1}{2}},$$

and let the characteristic direction associated with the operator $\phi\dfrac{\partial}{\partial t}+\mathbf{b}\cdot\nabla$ be denoted by y(x, t), so that

$$\frac{\partial}{\partial y}=\frac{\phi(x)}{\psi(x,t)}\frac{\partial}{\partial t}+\frac{\mathbf{b}(x,t)}{\psi(x,t)}\cdot\nabla.$$

Then (70) reduces to

$$\psi\frac{\partial S}{\partial y}+\nabla\cdot\left(K\,f_w(S)\,\lambda_o(S)\,\frac{dP_c}{dS}\,\nabla S\right)=q_1(P,S). \qquad (71)$$

We note that the characteristic direction y depends on the velocity **u**. Because the saturation step $t^{n-1,m}$ relates to pressure steps by $t^{n-1}<t^{n-1,m}\le t^n$, we need a velocity approximation for (71) based on $\mathbf{u}_h^{n-1}$ and earlier values. For this, we utilize a linear extrapolation approach:

If n ≥ 2, take the linear extrapolation of $\mathbf{u}_h^{n-2}$ and $\mathbf{u}_h^{n-1}$ determined by

$$\mathbf{E}(\mathbf{u}_h^{n-1,m})=\left(1+\frac{t^{n-1,m}-t^{n-1}}{t^{n-1}-t^{n-2}}\right)\mathbf{u}_h^{n-1}-\frac{t^{n-1,m}-t^{n-1}}{t^{n-1}-t^{n-2}}\mathbf{u}_h^{n-2}.$$

For n = 1, define

$$\mathbf{E}(\mathbf{u}_h^{0,m})=\mathbf{u}_h^0.$$

$\mathbf{E}(\mathbf{u}_h^{n-1,m})$ is first order accurate in time in the first pressure step and second order accurate in the later steps.

The MMOC is defined with periodic boundary conditions. For this reason, we assume that Ω is a rectangular domain, and all functions in (71) are spatially Ω-periodic. Let $V_h\subset H^1(\Omega)$ be any finite element space. Then an MMOC procedure for (71) is as follows:

For each $0\le n\le N$ and $1\le m\le M^{(n)}$, find $S_h^{n,m}\in V_h$ such that

$$\left\langle\phi\frac{S_h^{n,m}-\hat{S}_h^{n,m-1}}{t^{n,m}-t^{n,m-1}},w_h\right\rangle+\left\langle\mathbf{A}(S_h^{n,m-1})\,\nabla s_h^{n,m},\nabla w_h\right\rangle=\left\langle\tilde{q}_1(P_h^n,S_h^{n,m-1}),w_h\right\rangle,$$

$$w_h\in V_h, \qquad (72)$$

where

$$\mathbf{A}(S) = -\mathbf{K}\, f_w(S)\, \lambda_o(S)\, \frac{dP_c}{dS},$$

$$\hat{S}_h^{n,m-1} = S_h^{n,m-1}\left\langle \mathbf{x} - \frac{df_w}{dS}(S_h^{n,m-1})\frac{\mathbf{E}(\mathbf{u}_h^{n,m})}{\phi(x)}\Delta t^{n,m}, t^{n,m-1}\right\rangle,$$

with $\Delta t^{n,m} = t^{n,m} - t^{n,m-1}$. The initial approximate solution S_h^0 can be defined as any appropriate projection of S_0 in V_h. For the improved approach, the term $\left\langle \mathbf{A}(S_h^{n,m-1})\, \nabla s_h^{n,m}, \nabla w_h \right\rangle$ in (72) is replaced by $\left\langle \mathbf{A}(S_h^{n,m-1})\, \nabla S_h^{n,m-1}, \nabla w_h \right\rangle$.

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